Probability Theory
2015/16 Semester IIb
Instructor: Daniel Valesin
Final Exam
14/6/2016
Duration: 3 hours

Name: ______________________________
Student number: ____________

This exam contains 10 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page.

Your answers should be written in this booklet. Avoid handing in extra paper.

You are allowed to have two hand-written sheets of paper and a calculator.

You are required to show your work on each problem.

Do not write on the table below.

Problem	Points	Score
1	14	
2	14	
3	14	
4	14	
5	14	
6	7	
7	7	
8	6	
Total:	90	

Standard Normal cumulative distribution function
The value given in the table is $F_X(x)$ for $X \sim \mathcal{N}(0,1)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

1. (a) (7 points) A closet has 5 pairs of shoes. I open the closet in the dark and take 4 shoes at random, without replacement (that is: whenever I take a shoe, it is chosen uniformly among all shoes that are still inside the closet). Find the probability that I can form at least one pair with the shoes I took.

   (b) (7 points) Bas is doing a cycling tour of Morocco. Today he wakes up in Rabat and needs to travel to Casablanca. Before he starts, he can study a map and ask for directions. If he does both, the probability that he will get lost is 0.15. If he studies the map but does not ask for directions, the probability that he will get lost is 0.4. If he asks for directions but does not study the map, the probability that he will get lost is 0.3. If he does neither, the probability that he will get lost is 0.7.

   Suppose that he uses a fair coin toss to decide whether or not to ask for directions (heads $\rightarrow$ asks; tails $\rightarrow$ does not ask), and another independent fair coin toss to decide whether or not to study the map (heads $\rightarrow$ studies; tails $\rightarrow$ does not study).

   Given that he arrives in Casablanca without getting lost, what is the probability that he studied the map?

2. Let $\ldots, X_{-3}, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ be independent Bernoulli(p) random variables, $p \in (0,1)$. Define

$$Y_n = -1 + \min\{m \geq n : X_m \neq X_n\} - \max\{m \leq n : X_m \neq X_n\}, \qquad n \in \{\ldots, -1, 0, 1, \ldots\}.$$

   (a) (7 points) Find the probability mass function of Y_n.

   (b) (7 points) Find $\mathbb{E}(Y_n)$. You may use the formula:

$$\sum_{n=1}^{\infty} n^2 \cdot q^n = \frac{q+q^2}{(1-q)^3}, \quad 0 < q < 1.$$

3. We have n balls and m urns. The balls are placed in the urns one by one, and any given ball has the same probability of going into any urn. Let X be the number of urns that receive at least one ball.

   (a) (7 points) Find $\mathbb{E}(X)$.

   (b) (7 points) Find $\mathrm{Var}(X)$.

4. (a) (7 points) A random variable X has pdf

$$f_X(x) = \frac{c}{x^2 - x}, \qquad 2 < x < 5.$$

Find c and $\mathbb{E}(\lfloor X \rfloor)$ ($\lfloor \cdot \rfloor$ denotes the *floor function*. For a non-negative real number x, $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$).

(b) (7 points) Two random variables Y and Z have joint probability density function

$$f_{Y,Z}(y, z) = \frac{e^{-z}}{z}, \quad 0 < y < z < \infty.$$

Compute $\mathbb{E}(Y^2 \mid Z = z)$.

5. (a) (7 points) Prove that, if $X \sim \text{Binomial}(n, p)$, then the moment-generating function of X is $M_X(t) = (1 + p(e^t - 1))^n$.

   (b) (7 points) Prove that, if $Y \sim \text{Poisson}(\lambda)$, then the moment-generating function of Y is $M_Y(t) = e^{\lambda(e^t - 1)}$.

6. (7 points) If $X_1, X_2, \dots$ are independent Poisson(1) random variables, then for each n,

$$X_1 + \cdots + X_n \sim \text{Poisson}(n)$$

(you do not need to prove this). Use this fact to prove that

$$\text{for all } \varepsilon > 0, \qquad \sum_{i \in \mathbb{N}:\, (1-\varepsilon)n \le i \le (1+\varepsilon)n} \frac{n^i}{i!} \cdot e^{-n} \xrightarrow{n\to\infty} 1$$

and

$$\sum_{i=0}^{n} \frac{n^i}{i!} \cdot e^{-n} \xrightarrow{n\to\infty} \frac{1}{2}.$$

7. (7 points) Suppose $X_1, X_2, \ldots$ are Bernoulli(p) random variables and assume that, for some constant $K > 0$, we have

$$\mathrm{Cov}(X_i, X_j) = 0 \quad \text{for all } i, j \text{ with } |i - j| > K.$$

Show that $\frac{X_1 + \cdots + X_n}{n}$ converges in probability to p as $n \to \infty$.
Hint. First prove that $|\mathrm{Cov}(X_i, X_j)| \leq 1$ for every i, j. Then use this to obtain an upper bound for $\mathrm{Var}(\sum_{i=1}^{n} X_i)$. Finally, try to repeat the proof of the weak law of large numbers, using Chebychev's inequality.

8. (6 points) A person has 100 light bulbs whose lifetimes (in hours) are independent random variables following an Exponential(5) distribution (that is, with pdf $f(x) = \frac{1}{5}e^{-x/5}$ for $x > 0$). If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.